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Existence, decay and blow up of solutions for a Petrovsky equation with a fractional time delay term

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Abstract. In this paper, we consider a Petrovsky equation with fractional time delay term in a bounded domain. Firstly, we prove the existence of solutions using the semigroup theory. Later, we establish the decay of solutions. Finally, we obtain the blow up of the solutions.

1. Introduction

In this paper, we study the following Petrovsky equation with a fractional time delay term

$$
(1)\quad \begin{cases} u_{tt}+\Delta^2 u+\alpha_1\partial_t^{\alpha,\beta}u(t-\tau)+\alpha_2 u_t=|u|^{q-2}u, & x\in\Omega,\ t>0; \\ u(x,t)=\frac{\partial}{\partial \upsilon}u(x,t)=0, & x\in\partial\Omega,\ t>0; \\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega; \\ u_t(x,t-\tau)=f_0(x,t-\tau), & x\in\Omega,\ t\in(0,\tau); \end{cases}
$$

where Ω is a bounded domain of R^n, with a smooth boundary $\partial\Omega$, υ is the unit outer normal to $\partial\Omega$. α_1 and α_2 are positive real numbers such that $\alpha_1\beta^{\alpha-1}<\alpha_2$. The constants $q>2$ and $\tau>0$ is the time delay. Also, (u_0,u_1,f_0) the initial data belong to an appropriate function space. The notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative (see [4, 5, 20]) defined by the following formula

$$
\partial_t^{\alpha,\beta}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}u(\tau)\,d\tau,\quad 0<\alpha<1,\ \beta>0.
$$

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– The fourth-order equation has its origin in the canonical model introduced by Petrovsky [21,22]. This type of equations arises in many branches in sciences such as acoustics, optics, geophysics and ocean acoustics [6].
– Time delay appears in many practical problems such as economic, thermal, biological, physical, chemical phenomena and it can be a source of instability [11].
– Fractional derivatives and integrals arise naturally in physics, biology, chemistry, ecology (see [15, 20, 23]).

The decay phenomena commonly arise in solutions to the evolution equations of various types. Understanding the conditions under which such phenomena occur is of practical interest. There are several methods to show the decay of solutions. A recent comprehensive overview of these methods can be found in the monograph by Pişkin [18] and Straughan [24]. The blow up phenomena commonly arise in solutions to the evolution equations of various types. Understanding the conditions under which such phenomena occur is of practical interest. However, accurately computing the precise blow-up time is often challenging. Despite this challenge, it is still possible to estimate the blow-up time using various methods. A recent comprehensive overview of these methods can be found in the monograph by Al'shin et al. [1], Hu [8] and Pişkin [17].

Kirane and Tatar [13] considered the following equation

$$u_{tt} - \Delta u + \partial_t^{\alpha} u = |u|^{p-2} u.$$

They demonstrated the exponential growth for a fractionally damped wave equation.

Aounallah et al. [2] studied the following wave equation

$$u_{tt} - \Delta u + \alpha_1 \partial_t^{\alpha,\beta} u(t-\tau) + \alpha_2 u_t = |u|^{p-2} u.$$

They established the well-posedness, blow-up and asymptotic behaviour for a wave equation with a time delay condition of fractional type.

Pişkin and Uysal [19] studied the following equations

$$u_{tt} + \Delta^2 u + \partial_t^{1+\alpha} u = |u|^{p-1} u.$$

They proved the blow-up of solution.

Nicaise and Pignotti [10] considered as follows

$$u_{tt} - \Delta u + \alpha_1 u(t-\tau) + \alpha_2 u_t = f(u).$$

They demonstrated that the energy is exponentially stable when $\alpha_2 < \alpha_1$.

Kafini and Messaoudi [12] proved the following delayed wave equation with logarithmic source term

$$u_{tt} - \Delta u + \alpha_1 u(t-\tau) + \alpha_2 u_t = |u|^{p-2} u \ln |u|^k.$$

They investigated the local existence and blow-up of solutions.

Georgiev and Todorova [7] studied the following equations

$$u_{tt}-\Delta u+\alpha u_t\left|u_t\right|^{m-1}=bu\left|u\right|^{p-1}.$$

They considered the existence of a solution of the wave equation nonlinear damping and source term.

Our purpose of this paper is to study the local-global existence, decay and blow-up of solutions of the initial-boundary problem (1) in a bounded domain.

Our study is divided into six parts. In section 2, we give some important lemmas. In Section 3, we obtain the well-posedness by the semigroup theory. In Section 4, we prove the global existence results. In Section 5, we get decay of solutions. Finally, we establish the blow- up of solutions.

2. Preliminaries

In this part, we will restate problem (1), for which we need the following lemma.

Lemma 1 ([9]). *Set η is the function:*

$$\eta\left(\xi\right)=\left|\xi\right|^{\frac{2\alpha-1}{2}},\quad \xi\in R,\ 0<\alpha<1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\begin{cases}\phi_t\left(x,\xi,t\right)+\left(\xi^2+\beta\right)\phi\left(x,\xi,t\right)-U\left(x,t\right)\eta\left(\xi\right)=0,\ \xi\in R,\ t>0,\ \beta>0,\\ \phi\left(x,\xi,0\right)=0,\\ O\left(t\right)=\frac{\sin(\alpha\pi)}{\pi}\int_{-\infty}^{+\infty}\phi\left(x,\xi,t\right)\eta\left(\xi\right)d\xi\end{cases}$$

is given by

$$O=I^{1-\alpha,\beta}U,$$

here

$$I^{\alpha,\beta}u\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}e^{-\beta(t-\tau)}u\left(\tau\right)d\tau.$$

Lemma 2 ([3]). *If $\lambda\in D_\beta=\mathbb{C}\backslash\left(-\infty,-\beta\right)$ then*

$$\int_{-\infty}^{+\infty}\frac{\eta^2\left(\xi\right)}{\lambda+\beta+\xi^2}d\xi=\frac{\pi}{\sin\left(\alpha\pi\right)}\left(\lambda+\beta\right)^{\alpha-1}.$$

The damping and delay functions are considered under the following assumptions.

$$\alpha_1\beta^{\alpha-1}<\alpha_2. \tag{2}$$

Now, we introduce, as in [10], the new variable

$$z\left(x,\rho,t\right)=u_t\left(x,t-\rho\tau\right),\quad x\in\Omega,\ \rho\in\left(0,1\right),\ t\in R_+. \tag{3}$$

Then, we get

$$z_t\left(x,\rho,t\right)=\frac{-1}{\tau}z_\rho\left(x,\rho,t\right),\quad x\in\Omega,\ \rho\in\left(0,1\right),\ t\in R_+. \tag{4}$$

Also, by (3)-(4) and applying Lemma 1, we can reformulate problem (1) as the following system

$$(5)\quad \begin{cases} u_{tt}+\Delta^2 u+b\int_{-\infty}^{+\infty}\phi(x,\xi,t)\,\eta(\xi)\,d\xi+\alpha_2 u_t=|u|^{q-2}u, & x\in\Omega,\ t>0,\\ \phi_t(x,\xi,t)+\left(\xi^2+\beta\right)\phi(x,\xi,t)-z(x,1,t)\,\eta(\xi)=0, & x\in\Omega,\ \xi\in R,\ t>0,\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & x\in\Omega,\ \rho\in(0,1),\ t>0,\\ u(x,t)=\frac{\partial}{\partial\nu}u(x,t)=0, & x\in\partial\Omega,\ t>0,\\ z(x,0,t)=u_t(x,t), & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & x\in\Omega,\\ \phi(x,\xi,0)=0, & x\in\Omega,\ \xi\in R,\\ z(x,\rho,0),=f_0(x,-\rho\tau), & x\in\Omega,\ \rho\in(0,1), \end{cases}$$

here $b=\frac{\sin(\alpha\pi)}{\pi}\alpha_1$.

Lemma 3. *Assume that* $z\in L^2(\Omega)$ *and* $\xi\phi\in L^2(\Omega\times(-\infty,+\infty))$ *hold. Then*

$$\left|\int_\Omega z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx\right|$$
$$\leq A_0\int_\Omega|z(x,\rho,t)|^2dx+\frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2d\xi dx$$

for a positive constant A_0.

Proof. Thanks to the Cauchy-Schwarz inequality, we have

$$\left|\int_{-\infty}^{+\infty}\eta(\xi)\,\phi(x,\xi,t)\,d\xi dx\right|$$
$$\leq\left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2d\xi\right)^{\frac{1}{2}}.$$

By using Young's inequality, we get

$$\left|\int_\Omega z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi(x,\xi,t)\,d\xi dx\right|$$
$$\leq A_0\int_\Omega|z(x,\rho,t)|^2dx+\frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2d\xi dx$$

with

$$A_0=\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}d\xi.$$

This completes the proof. □

Now, we define the energy funcional of the problem (5) by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{q}\|u\|_q^q + s\tau\int_\Omega\int_0^1|z(x,\rho,t)|^2 d\rho dx, \tag{6}$$

where s is a positive constant verifying

$$bA_0 < s < \alpha_2 - bA_0. \tag{7}$$

Lemma 4. *Suppose that* (2) *holds and*

$$\begin{cases} 2 < q < \infty, & \text{if } n = 1,2,3,4; \\ 2 < q \leq \frac{2n}{n-4}, & \text{if } n \geq 5. \end{cases} \tag{8}$$

Then, the energy functional defined by (6) *satisfies*

$$\frac{dE(t)}{dt} \leq -C\int_\Omega\left(|z(x,1,t)|^2 + |z(x,0,t)|^2\right)dx - \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2 d\xi dx, \tag{9}$$

for a positive constant C.

Proof. Multiply u_t with the first equation of (5) and integrating by parts over Ω, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{q}\|u\|_q^q\right] + \alpha_2\|u_t\|_2^2 + b\int_\Omega u_t\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx = 0. \tag{10}$$

Multiply $b\phi$ with the second equation of (5) and integrating over $\Omega \times (-\infty,+\infty)$, we obtain

$$\frac{d}{dt}\left\{\frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx\right\} + b\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2 d\xi dx - b\int_\Omega z(x,1,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx = 0. \tag{11}$$

Multiply $2sz$ with the third equation of (5) and integrating over $\Omega \times (0,1)$, we obtain

$$\frac{d}{dt}\left\{\tau s\int_\Omega\int_0^1|z(x,\rho,t)|^2 d\rho dx\right\} + s\int_\Omega\left[|z(x,1,t)|^2 - |z(x,0,t)|^2\right]dx = 0. \tag{12}$$

By summing (10), (11), (12) and using $u_t = z(x,0,t)$, we get

$$\begin{aligned}\frac{dE(t)}{dt} = &-(\alpha_2 - s)\int_\Omega |z(x,0,t)|^2\,dx - s\int_\Omega |z(x,1,t)|^2 \\ &- b\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2\,d\xi dx \\ &- b\int_\Omega z(x,0,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi(x,\xi,t)\,d\xi dx \\ &+ b\int_\Omega z(x,1,t)\int_{-\infty}^{+\infty}\eta(\xi)\,\phi(x,\xi,t)\,d\xi dx.\end{aligned}$$

By using Lemma 3, we have

$$\begin{aligned}\frac{dE(t)}{dt} \le &-C\int_\Omega\left(|z(x,1,t)|^2 + |z(x,0,t)|^2\right)dx \\ &- \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2\,d\xi dx\end{aligned}$$

with

$$C = \min\left\{(s - bA_0),(\alpha_2 - bA_0 - s)\right\}.$$

Given that s is selected in accordance with assumption (7), the constant C turns out to be positive. This concludes the proof. □

3. Well-posedness

Let us define $v = u_t$ and introduce the vector

$$U = \begin{pmatrix} u \\ v \\ \phi \\ z \end{pmatrix},$$

with the initial condition specified by

$$U(0) = U_0 = \begin{pmatrix} u_0 \\ u_1 \\ 0 \\ f_0(.,-\rho\tau) \end{pmatrix}.$$

We also define the nonlinear operator $J(U)$ as

$$J(U) = \begin{pmatrix} 0 \\ |u|^{q-2}\,u \\ 0 \\ 0 \end{pmatrix}.$$

Then (5) can be rewritten as an abstract problem

$$\begin{cases} U_t + \mathcal{A}U = J\left(U\left(t\right)\right), \\ U_0\left(0\right) = \left(u_0, u_1, 0, f_0(., -\rho\tau)\right)^T, \end{cases} \tag{13}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} -v \\ \Delta^2 u + b\int_{-\infty}^{+\infty} \phi\left(x, \xi, \right) \eta\left(\xi\right) d\xi + \alpha_2 v \\ \left(\xi^2 + \beta\right) \phi\left(x, \xi\right) - z\left(x, 1\right) \eta\left(\xi\right) \\ \frac{1}{\tau} z_\rho\left(x, \rho\right) \end{pmatrix}$$

with domain

$$D\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} : u \in H^2(\Omega),\ v \in H_0^1\left(\Omega\right),\ z_\rho \in L^2\left(\Omega \times (0,1)\right), \\ z(.,0,.) = v,\ \xi\phi \in L^2\Omega \times \left(-\infty, +\infty\right), \\ \left(\xi^2 + \beta\right) \phi - z\left(x, 1, t\right) \eta\left(\xi\right) \in L^2\left(\Omega \times \left(-\infty, +\infty\right)\right) \end{array} \right\},$$

where the space $\mathcal{H}$ is defined by:

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2\big(\Omega \times (-\infty, +\infty)\big) \times L^2\big(\Omega \times (0,1)\big)$$

equipped with the inner product

$$\left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}} = \int_\Omega \left[\Delta u \Delta \widetilde{u} + v\widetilde{v}\right] dx + b\int_\Omega \int_{-\infty}^{+\infty} \phi(x,\xi)\widetilde{\phi}(x,\xi) d\xi dx$$
$$+ 2s\tau \int_\Omega \int_0^1 z(x,\xi)\widetilde{z}(x,\xi) dx d\rho.$$

Theorem 1. *Suppose that* (7) *and* (8) *hold. Then for any* $U_0 \in \mathcal{H}$*, problem* (13) *has a local unique weak solution*

$$U \in C\big([0,T), \mathcal{H}\big).$$

Proof. Following the approach in [14, 16], we demonstrate that the operator $\mathcal{A}$ is maximal monotone and the function J is a locally Lipschitz continuous. Initially, for every $U \in D(\mathcal{A})$,by applying inequalities (13) and (9), we obtain

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq C\int_\Omega \left[\left|z\left(x,1\right)\right|^2 + \left|z\left(x,0\right)\right|^2\right] dx$$
$$+ \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) \left|\phi\left(x,\xi\right)\right|^2 d\xi dx.$$

This inequality confirms that $\mathcal{A}$ is a monotone operator.

To establish maximality, we aim to show that the operator $I + \mathcal{A}$ is onto. Specifically, for any given $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we need to find a $U = (u, v, \phi, z)^T \in D(\mathcal{A})$ such that

$$\left(I + \mathcal{A}\right) U = F.$$

Namely,

$$\begin{cases} u - v = f_1(x), \\ (1+\alpha_2)u + \Delta^2 u + b\int_{-\infty}^{+\infty}\phi(x,\xi,)\eta(\xi)d\xi = f_2(x), \\ \phi(x,\xi) + \left(\xi^2+\beta\right)\phi(x,\xi) - z(x,1)\eta(\xi) = f_3(x,\xi), \\ z(x,\rho) + \frac{1}{\tau}z_\rho(x,\rho) = f_4(x,\rho). \end{cases} \tag{14}$$

Provided that u exhibits sufficient regularity, one can deduce from the first and third equations in (14) that

$$v = u - f_1 \tag{15}$$

and

$$\phi(x,\xi) = \frac{f_3(x,\xi) + z(x,1)\eta(\xi)}{\xi^2+\beta+1}, \quad x\in\Omega,\ \xi\in R. \tag{16}$$

Conversely, the fourth equation in (14), subject to the initial condition $z(x,0) = u - f_1$ admits a unique solution given by

$$z(x,\rho) = (u - f_1(x))\, e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau\sigma} f_4(x,\sigma)d\sigma,\ x\in\Omega, \rho\in(0,1). \tag{17}$$

Substituting (15) in the second equation of (14), we have

$$(1+\alpha_2)u + \Delta^2 u + b\int_{-\infty}^{+\infty}\phi(x,\xi,)\eta(\xi)d\xi = f_2(x) + (1+\alpha_2)f_1(x). \tag{18}$$

Solving equation (18) is equivalent to finding $u\in H^2(\Omega)$ such that

$$\begin{aligned} &\int_\Omega\left[(1+\alpha_2)u + \Delta^2 u\right]w dx + b\int_\Omega w\int_{-\infty}^{+\infty}\phi(x,\xi)\eta(\xi)d\xi dx \\ &= \int_\Omega\left[f_2(x) + (1+\alpha_2)f_1(x)\right]w dx, \quad w\in H_0^1(\Omega). \end{aligned} \tag{19}$$

By using (19), (17) and (16), we get

$$\begin{aligned} \int_\Omega\left(\mu u + \Delta^2 u\right)w = &\int_\Omega\left(f_2(x) + \mu f_1(x)\right)w dx \\ &- b\int_\Omega w\int_{-\infty}^{+\infty}\frac{\eta(\xi)f_3(x,\xi)}{\xi^2+\beta+1}d\xi dx \\ &- b\tau e^{-\tau}A_1\int_\Omega w\int_0^1 e^{\tau\sigma}f_4(x,\sigma)d\sigma dx,\ \ w\in H_0^1(\Omega), \end{aligned} \tag{20}$$

here

$$\mu = 1+\alpha_2+be^{-\tau}A_1 > 0, \qquad A_1 = \int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta+1}d\xi.$$

As a result, problem (20) is equivalent to the problem

$$B(u,w) = L(w), \tag{21}$$

here the bilinear form $B : H_0^2(\Omega) \times H_0^2(\Omega) \to R$ defined by

$$B(u,w) = \mu \int_\Omega uw dx + \int_\Omega \Delta u \Delta w dx$$

and the linear form $L : H_0^2(\Omega) \to R$ by

$$L(w) = \int_\Omega \left(f_2(x) + \mu f_1(x)\right) w dx - b \int_\Omega w \int_{-\infty}^{+\infty} \frac{\eta(\xi) f_3(x,\xi)}{\xi^2 + \beta + 1} d\xi dx$$
$$- b\tau e^{-\tau} A_1 \int_\Omega w \int_0^1 e^{\tau\sigma} f_4(x,\sigma) d\sigma dx.$$

It is straightforward to verify that B is coercive and continous and L is continous. So, applying the Lax-Milgram theorem, we deduce that for all $w \in H_0^2(\Omega)$ problem (21) admits a unique solution $u \in H_0^2(\Omega)$. Applying the classical eliptic regularity, it follows from (21) that $u \in H_0^2(\Omega)$. Using the second equation of (14) and Green's formula, we have

$$\int_\Omega \left[(1+\alpha_2) u + \Delta^2 u + b \int_{-\infty}^{+\infty} \phi(\xi)\eta(\xi) d\xi - f_2 \right] w = 0, \quad w \in H_0^1(\Omega).$$

Hence,

$$(1+\alpha_2) u + \Delta^2 u + b \int_{-\infty}^{+\infty} \phi(x,\xi) \eta(\xi) d\xi = f_2(x) \in L^2(\Omega).$$

Using the third equation of (14), we get

$$\int_\Omega \int_{-\infty}^{+\infty} \left[\phi(\xi) + \left(\xi^2 + \beta\right) \phi(\xi) - z(1) \eta(\xi) - f_3(\xi) \right] w d\xi = 0, \quad w \in H_0^1(\Omega).$$

Hence,

$$\phi(x,\xi) + \left(\xi^2 + \beta\right) \phi(x,\xi) - z(x,1) \eta(\xi) = f_3(x,\xi) \in L^2(\Omega \times (-\infty, +\infty)).$$

Therefore,

$$U \in D(\mathcal{A}).$$

Consequently, $I + \mathcal{A}$ is an onto operator.

To conclude, we show that the mapping $J : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. For any $U \in \mathcal{H}$, we observe that

$$\begin{aligned} \left\| J(U) - J(\widetilde{U}) \right\|_{\mathcal{H}}^2 &= \left\| 0, u|u|^{q-2} - \widetilde{u}|\widetilde{u}|^{q-2}, 0, 0 \right\|_{\mathcal{H}}^2 \\ &= \left\| u|u|^{q-2} - \widetilde{u}|\widetilde{u}|^{q-2} \right\|^2. \end{aligned}$$

It is easily to verify that

$$\left\| u|u|^{q-2} - \widetilde{u}|\widetilde{u}|^{q-2} \right\|^2 \le C \|u - \widetilde{u}\|_{H_0^1(\Omega)}^2.$$

Hence, J satisfies the local Lipschitz condition. □

4. Global existence

In this section, we establish the global existence of solutions. To begin, we introduce the following two functionals

$$I(t) = b\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2\, d\xi dx + \|\Delta u\|^2 - \|u\|_q^q + s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2\, d\rho dx \tag{22}$$

and

$$J(t) = \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2\, d\xi dx + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{q}\|u\|_q^q + s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2\, d\rho dx. \tag{23}$$

By the definition $J(t)$ and $E(t)$, we have

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(t). \tag{24}$$

Lemma 5. *Suppose that* (2) *and* (8) *hold. Then, for* $U_0 \in \mathcal{H}$ *satisfying*

$$\begin{cases} \overline{\beta} = C_*^q \left(\frac{2q}{(q-2)}E(0)\right)^{\frac{q-2}{2}} < 1, \\ I(0) > 0. \end{cases} \tag{25}$$

Then

$$I(t) > 0, \quad \text{for all } t > 0.$$

Proof. Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that

$$I(t) \geq 0, \quad \text{for all } t \in [0, T^*].$$

By (22) and (23), we have

$$\begin{aligned} \frac{2q}{q-2}J(t) &= b\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2\, d\xi dx + \frac{2}{q-2}I(t) \\ &\quad + \|\Delta u\|^2 + \frac{2(q-1)s\tau}{q-2}\int_\Omega \int_0^1 |z(x,\rho,t)|^2\, d\rho dx \\ &\geq \|\Delta u\|^2. \end{aligned} \tag{26}$$

Thus, by (9), (24) and (26), we deduce that

$$\|\Delta u(t)\|^2 \leq \frac{2q}{(q-2)}E(t) \leq \frac{2q}{(q-2)}E(0)$$

for all $t \in [0, T^*]$. Thanks to Sobolev-Poincare inequality and (25), we have

$$\begin{aligned}\|u\|_q^q &\leq C_*^q \|\Delta u\|_2^q \\ &\leq C_*^q \left(\frac{2q}{(q-2)} E(0)\right)^{\frac{q-2}{2}} \|\Delta u\|^2 \\ &< \|\Delta u\|^2, \quad \text{for all } t \in [0, T^*].\end{aligned}$$

After this

$$\begin{aligned}I(t) = b \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + \|\Delta u\|^2 \\ - \|u\|_q^q + s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx > 0, \quad \text{for all } t \in [0, T^*].\end{aligned}$$

By iterating this process and utilizing the inequality

$$\lim_{t\to T^*} C_*^q \left(\frac{2q}{(q-2)} E(0)\right)^{\frac{q-2}{2}} < 1,$$

we can take $T^* = T$. □

Theorem 2. *Suppose that* (7) *and* (8) *hold, and* $U_0 \in D(\mathcal{A})$ *satisfying* (25). *Then the solution of system* (5) *is global and bounded.*

Proof. It suffices to show that $\|u_t\|^2 + \|\Delta u\|^2$ is bounded independently of t. We get from (24) and (26)

$$\begin{aligned}E(0) \geq E(t) &= \frac{1}{2}\|u_t\|^2 + J(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{(q-2)}{2q}\|\Delta u\|^2.\end{aligned}$$

Therefore,

$$\|u_t\|^2 + \|\Delta u\|^2 \leq \xi_1 E(0),$$

where ξ_1 is a positive constant, which depends only on the parameter q. □

5. Decay

In this part, we prove the decay estimates of energy to the problem (5). For $N > 0$ and $\varepsilon_1 > 0$, we define a perturbed modified energy by

$$L(t) = NE(t) + \varepsilon_1 K_1(t) + K_2(t)$$

where

$$\begin{aligned}K_1(t) &= \int_\Omega u_t u dx + \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |M(x,\xi,t)|^2 d\xi dx, \\ K_2(t) &= \tau \int_\Omega \int_0^1 e^{-\tau\rho} |z(x,\rho,t)|^2 d\rho dx,\end{aligned} \tag{27}$$

and

$$M(x,\xi,t) = \int_0^t \phi(x,\xi,\sigma) d\sigma - \frac{\tau\eta(\xi)}{\xi^2+\beta}\int_0^1 f_0(x,-\rho\tau) d\rho + \frac{u_0(x)\eta(\xi)}{\xi^2+\beta}.$$

Lemma 6. *Let (u,ϕ,z) be regular solution of the problem* (5), *then*

$$\begin{aligned}
&\int_{\Omega}\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)\phi\left(x,\xi,t\right)M\left(x,\xi,t\right)d\xi dx\\
&=\int_{\Omega}u\left(x,t\right)\int_{-\infty}^{+\infty}\phi\left(x,\xi,t\right)\eta(\xi)d\xi dx\\
&\quad-\tau\int_{\Omega}\int_{0}^{1}z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi d\rho dx\\
&\quad-\int_{\Omega}\int_{-\infty}^{+\infty}\left|\phi(x,\xi,t)\right|^2 d\xi dx.
\end{aligned}\tag{28}$$

Proof. Using the second equation in (5), we have

$$\begin{aligned}
\left(\xi^2+\beta\right)\phi\left(x,\xi,t\right)&=z\left(x,1,t\right)\eta\left(\xi\right)-\phi_t\left(x,\xi,t\right)\\
&=\eta\left(\xi\right)\left[z\left(x,1,t\right)-z\left(x,0,t\right)\right]\\
&\quad+u_t\left(x,t\right)\eta\left(\xi\right)-\phi_t\left(x,\xi,t\right).
\end{aligned}$$

Observe that

$$-\tau\int_0^1 z_t\left(x,\rho,t\right)d\rho=\int_0^1 z_\rho\left(x,\rho,t\right)d\rho=z\left(x,1,t\right)-z\left(x,0,t\right).$$

After this

$$\begin{aligned}
\left(\xi^2+\beta\right)\phi\left(x,\xi,t\right)&=-\tau\eta\left(\xi\right)\int_0^1 z_t\left(x,\rho,t\right)d\rho\\
&\quad+u_t\left(x,t\right)\eta\left(\xi\right)-\phi_t\left(x,\xi,t\right).
\end{aligned}$$

Integrating the last equation over $[0,t]$, we obtain

$$\begin{aligned}
\int_0^t\left(\xi^2+\beta\right)\phi\left(x,\xi,\sigma\right)d\sigma&=-\tau\eta\left(\xi\right)\int_0^1 z\left(x,\rho,t\right)d\rho\\
&\quad+\tau\eta\left(\xi\right)\int_0^1 f_0\left(x,-\rho\tau\right)d\rho\\
&\quad+u\left(x,t\right)\eta\left(\xi\right)-u_0\left(x\right)\eta(\xi)-\phi(x,\xi,t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\xi^2+\beta\right)M\left(x,\xi,t\right)&=-\tau\eta\left(\xi\right)\int_0^1 z\left(x,\rho,t\right)d\rho\\
&\quad+u\left(x,t\right)\eta\left(\xi\right)-\phi\left(x,\xi,t\right).
\end{aligned}\tag{29}$$

Multiplying (29) by ϕ and integrating over $\Omega\times(-\infty,+\infty)$, we get (28). $\square$

Lemma 7. *Let (u, ϕ, z) be regular solution of the problem (5), then*

$$\begin{aligned}
&\left| \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) \left|M(x,\xi,t)\right|^2 d\xi dx \right| \\
&\leq 3\tau^2 A_0 \int_\Omega \int_0^1 \left|z(x,\rho,t)\right|^2 d\rho dx + 3A_0 C_*^2 \left\|\Delta u\right\|^2 \\
&+ \frac{3}{\beta} \int_\Omega \int_{-\infty}^{+\infty} \left|\phi(x,\xi,t)\right|^2 d\xi dx.
\end{aligned} \tag{30}$$

Proof. Invoking (29), to obtain

$$\begin{aligned}
&\left| \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) \left|M(x,\xi,t)\right|^2 d\xi dx \right| \\
&\leq \tau^2 A_0 \int_\Omega \left(\int_0^1 z(x,\rho,t) \right)^2 dx \\
&+ A_0 \left\|u\right\|^2 + \int_\Omega \int_{-\infty}^{+\infty} \frac{\left|\phi(x,\xi,t)\right|^2}{\xi^2 + \beta} d\xi dx \\
&+ 2 \int_\Omega \int_{-\infty}^{+\infty} \frac{\left|\phi(x,\xi,t)\, u(x,t)\, \eta(\xi)\right|}{\xi^2 + \beta} d\xi dx \\
&2\tau A_0 \int_\Omega \left| u(x,t) \int_0^1 z(x,\rho,t)\, d\rho \right| dx \\
&+ 2\tau \int_\Omega \int_{-\infty}^{+\infty} \frac{\left|\phi(x,\xi,t)\, \eta(\xi) \int_0^1 z(x,\rho,t)\, d\rho\right|}{\xi^2 + \beta} d\xi dx.
\end{aligned} \tag{31}$$

Next, we aim to bound the right-hand side of equation (31). Applying Hölder's inequality gives us

$$\int_0^1 z(x,\rho,t)\, d\rho \leq \left(\int_0^1 \left|z(x,\rho,t)\right|^2 d\rho \right)^{\frac{1}{2}}. \tag{32}$$

To estimate the fourth and fifth terms, we apply Young's inequality, obtaining

$$\begin{aligned}
&\int_\Omega \int_{-\infty}^{+\infty} \frac{\left|\phi(x,\xi,t) u(x,t)\, \eta(\xi)\right|}{\xi^2 + \beta} d\xi dx \\
&\leq \frac{A_0}{2} \left\|u\right\|^2 + \frac{1}{2} \int_\Omega \int_{-\infty}^{+\infty} \frac{\left|\phi(x,\xi,t)\right|^2}{\xi^2 + \beta} d\xi dx
\end{aligned}$$

and

$$\tau \int_\Omega \left| u(x,t) \int_0^1 z(x,\rho,t)\, d\rho \right| dx \leq \frac{\tau^2}{2} \int_\Omega \int_0^1 \left|z(x,\rho,t)\right|^2 d\rho dx + \frac{1}{2} \left\|u\right\|^2.$$

For the final term, applying Young's inequality, (32) along with Lemma 3 yields

$$\begin{aligned}&\tau\int_{\Omega}\int_{-\infty}^{+\infty}\frac{\left|\phi\left(x,\xi,t\right)\eta\left(\xi\right)\int_0^1 z\left(x,\rho,t\right)d\rho\right|}{\xi^2+\beta}d\xi dx\\ &\leq\frac{\tau^2 A_0}{2}\int_{\Omega}\int_0^1\left|z\left(x,\rho,t\right)\right|^2 d\rho dx\\ &\quad+\frac{1}{2}\int_{\Omega}\int_{-\infty}^{+\infty}\frac{\left|\phi\left(x,\xi,t\right)\right|^2}{\xi^2+\beta}d\xi dx.\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}&\left|\int_{\Omega}\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)\left|M\left(x,\xi,t\right)\right|^2 d\xi dx\right|\\ &\leq 3\tau^2\int_{\Omega}\int_0^1\left|z\left(x,\rho,t\right)\right|^2 d\rho dx+3A_0\left\|u\right\|^2\\ &\quad+3\int_{\Omega}\int_{-\infty}^{+\infty}\frac{\left|\phi\left(x,\xi,t\right)\right|^2}{\xi^2+\beta}d\xi dx.\end{aligned}$$

Using the fact that $\frac{1}{\xi^2+\beta}\leq\frac{1}{\beta}$ and Poincaré's inequality, then (30) is established. □

Lemma 8. *For ε_1 small and N large enough, we have*

$$\frac{N}{2}E(t)\leq L(t)\leq 2NE(t),\quad \textit{for all } t\geq 0. \tag{33}$$

Proof. Using Young's inequality and Poincaré's inequality, we get

$$\begin{aligned}L\left(t\right)\leq NE\left(t\right)&+\frac{\varepsilon_1}{2}\left\|u_t\right\|^2+\tau\int_{\Omega}\int_0^1 e^{-\tau\rho}\left|z\left(x,\rho,t\right)\right|^2 d\rho dx\\ &+\frac{\varepsilon_1 C_*^2}{2}\left\|\Delta u\right\|^2+\frac{b\varepsilon_1}{2}\int_{\Omega}\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)\left|M\left(x,\xi,t\right)\right|^2 d\xi dx.\end{aligned}$$

Using (6) and Lemma 7, we have

$$\begin{aligned}L\left(t\right)\leq&\frac{1}{2}\left\{N+\varepsilon_1\right\}\left\|u_t\right\|^2-\frac{N}{q}\left\|u\right\|_q^q\\ &+\tau\left\{Ns+\frac{3s\tau\varepsilon_1}{2}\right\}\int_{\Omega}\int_0^1\left|z\left(x,\rho,t\right)\right|^2 d\rho dx\\ &+\tau\int_{\Omega}\int_0^1 e^{-\tau\rho}\left|z\left(x,\rho,t\right)\right|^2 d\rho dx\\ &+\frac{1}{2}\left(N+\varepsilon_1 C_*^2\left\{1+3s\right\}\right)\left\|\Delta u\right\|^2\\ &+\frac{b}{2}\left(N+\frac{3\varepsilon_1}{\beta}\right)\int_{\Omega}\int_{-\infty}^{+\infty}\left|\phi\left(x,\xi,t\right)\right|^2 d\xi dx.\end{aligned}$$

So, by using (22), we have

$$
\begin{aligned}
2NE(t) - L(t) \geq & \frac{1}{2}\{N-\varepsilon_1\}\|u_t\|^2 + \frac{N}{q}I(t) \\
& + \tau\left\{\frac{(q-1)sN}{q} - e^{-\tau} - \frac{3s\tau\varepsilon_1}{2}\right\}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\, d\rho dx \\
& + \frac{1}{2}\left\{\frac{(q-2)N}{q} - \varepsilon_1 C_*^2\{1+3s\}\right\}\|\Delta u\|^2 \\
& + \frac{b}{2}\left\{\frac{(q-2)N}{q} - \frac{3\varepsilon_1}{\beta}\right\}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\, d\xi dx.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
L(t) - \frac{N}{2}E(t) \geq & \frac{1}{2}\left\{\frac{N}{2}-\varepsilon_1\right\}\|u_t\|^2 + \frac{N}{2q}I(t) \\
& + \tau\left\{\frac{(q-1)sN}{2q} - e^{-\tau} - \frac{3s\tau\varepsilon_1}{2}\right\}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\, d\rho dx \\
& + \frac{1}{2}\left\{\frac{(q-2)N}{2q} - \varepsilon_1 C_*^2\{1+3s\}\right\}\|\Delta u\|^2 \\
& + \frac{b}{2}\left\{\frac{(q-2)N}{q} - \frac{3\varepsilon_1}{\beta}\right\}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\, d\xi dx.
\end{aligned}
$$

By fixing ε_1 small and N large enough, we obtain $L(t) - \frac{N}{2}E(t) \geq 0$ and $2NE(t) - L(t) \geq 0$. The proof is completed. □

Lemma 9. *Suppose that* (1) *and* (6) *hold. Then, the functional* K_1 *defined by* (27) *satisfies*

$$
\begin{aligned}
K_1'(t) = & C_1\|u_t\|^2 - \frac{1}{2}\|\Delta u\|^2 - b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\, d\xi dx \\
& + \frac{b}{4}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2\, d\xi dx \\
& + \tau^2 s\int_\Omega\int_0^1|z(x,\rho,t)|^2\, d\rho dx + \|u\|_q^q,
\end{aligned}
\tag{34}
$$

for some positive constant C_1.

Proof. A direct differentiation of K_1, using Lemma 6, gives

$$
\begin{aligned}
K_1'(t) &= \|u_t\|^2 + \int_\Omega u u_{tt}\, dx + b\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)\phi(x,\xi,t)\, M(x,\xi,t)\, d\xi dx \\
&= \|u_t\|^2 - \|\Delta u\|^2 - b\tau\int_\Omega\int_0^1 z(x,\rho,t)\int_\Omega\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)\, d\xi d\rho dx \\
&\quad + \|u\|^2 - \int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2\, d\xi dx - \alpha_2\int_\Omega u_t u\, dx.
\end{aligned}
$$

Thanks to Young's inequality and Lemma 3, we obtain

$$\begin{aligned} K_1'(t) &\leq (1+\eta_1\alpha_2)\|u_t\|^2 - \left(1-\frac{\alpha_2 C_2^*}{4\eta_1}\right)\|\Delta u\|^2 \\ &+ \frac{b}{4}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\beta\right)|\phi(x,\xi,t)|^2 d\xi dx \\ &+ \tau^2 s\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx + \|u\|_q^q \\ &- b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx. \end{aligned}$$

By choosing $\eta_1 = \frac{\alpha_2 C_2^*}{2}$, then (34) is established. □

Lemma 10. *Assume that* (1) *and* (6) *hold. Then the functional* K_2 *and using the third equation in* (5)*, we have*

$$K_2'(t) \leq -\tau e^{-\tau}\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx + \|u_t\|_q^q. \tag{35}$$

Proof. By differentiating K_2 with respect to time and applying the third equation from reference (5), we obtain

$$\begin{aligned} K_2'(t) &= -2\tau\int_\Omega\int_0^1 e^{-\tau\rho} z(x,\rho,t)\, z_t(x,\rho,t)\, d\rho dx \\ &= -2\int_\Omega\int_0^1 e^{-\tau\rho} z(x,\rho,t)\, z_\rho(x,\rho,t)\, d\rho dx \\ &= -\int_\Omega\int_0^1 \frac{d}{d\rho}\left[e^{-\tau\rho}|z(x,\rho,t)|^2\right] d\rho dx \\ &- \tau\int_\Omega\int_0^1 e^{-\tau\rho}|z(x,\rho,t)|^2 d\rho dx \\ &= -\tau\int_\Omega\int_0^1 e^{-\tau\rho}|z(x,\rho,t)|^2 d\rho dx \\ &- e^{-\tau}\int_\Omega |z(x,1,t)|^2 dx + \|u_t\|^2. \end{aligned}$$

Then (35) is established. □

Theorem 3. *Assume that* (1) *and* (6) *hold, and* $U_0 \in \mathcal{H}$ *satisfying* (25). *Then any solution of* (5) *satisfies*

$$E(t) \leq Ke^{-wt}, \quad t \geq 0,$$

for some positive constants K *and* w *independent of* t.

Proof. By using (34) and (35), we get, for all $t \geq 0$,

$$\begin{aligned} L'(t) \leq & -(NC - C_1\varepsilon_1 - 1)\|u_t\|_2^2 - \frac{\varepsilon_1}{2}\|\Delta u\|^2 \\ & + \varepsilon_1 \|u\|_q^q - b\varepsilon_1 \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx \\ & - \frac{b}{2}\left(N - \frac{\varepsilon_1}{2}\right) \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2 d\xi dx \\ & - \tau\left(e^{-\tau} - s\tau\varepsilon_1\right) \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{36}$$

At this stage, we select ε_1 sufficiently small to ensure that

$$e^{-\tau} - s\tau\varepsilon_1 > 0,$$

and subsequently choose N large enough to satisfy the condition

$$N > \max\left\{\frac{C_1\varepsilon_1 + 1}{C}, 2\varepsilon_1\right\}.$$

Consequently, from the above, we deduce that there exist a positive constant m such that (36) becomes

$$L'(t) \leq -mE(t), \quad \text{for all } t \geq 0.$$

By using (33), we have

$$L'(t) \leq -wL(t), \quad \text{for all } t \geq 0. \tag{37}$$

A simple integration of (37) over $(0,t)$ leads to

$$L(t) \leq L(0)e^{-wt}, \quad t \geq 0.$$

As $L(t)$ and $E(t)$ are equivalent, we have

$$E(t) \leq ke^{-wt}, \quad t \geq 0.$$ □

6. Blow up

In this part, we prove the blow up of the solution of problem (5). Let (u,ϕ,z) be solution of (5) and define

$$\begin{aligned} H(t) = -E(t) = & \frac{1}{q}\|u\|_q^q - \frac{1}{2}\|u_t\|^2 - \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx \\ & - \frac{1}{2}\|\Delta u\|^2 - s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{38}$$

Lemma 11. *Assume that* (6) *holds. Then there exists a positive constant* $C_2 > 1$, *depending on* Ω *only, such that*

$$\|u\|_q^l \le C_2 \left[\|u\|_q^q + \|\Delta u\|^2\right]$$

for any $u \in H_0^1(\Omega)$ *and* $2 \le l \le q$.

Proof. If $\|u\|_q \le 1$, then $\|u\|_q^l \le \|u\|_q^2 \le C_* \|\Delta u\|^2$ by Sobolev embedding theorems.

If $\|u\|_q \ge 1$, then $\|u\|_q^l \le \|u\|_q^q$. This leads to the final result. □

Theorem 4. *Suppose the hypotheses of Theorem 1 are satisfied. Additionally, assume that*

$$E(0) \le 0.$$

Then the solution of system (5) *blows up in finite time.*

Proof. From (38) we get

$$H'(t) = -E'(t) \ge \frac{b}{2} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2 \, d\xi dx, \tag{39}$$

hence

$$0 < H(0) \le H(t) \le \frac{1}{q} \|u\|_q^q. \tag{40}$$

We then define

$$\varphi(t) = H^{1-\gamma}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\alpha_2 \varepsilon}{2} \|u\|^2, \tag{41}$$

for $\varepsilon > 0$ small to be chosen later and

$$0 < \gamma < \frac{q-2}{2q}. \tag{42}$$

By taking a derivative of (41) and using equation (5), we get

$$\begin{aligned} \varphi'(t) &= (1-\gamma) H^{-\gamma} H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 \\ &\quad - b\varepsilon \int_\Omega u \int_{-\infty}^{+\infty} \eta(\xi) \phi(x,\xi,t) d\xi dx + \varepsilon \|u\|_q^q. \end{aligned} \tag{43}$$

Using Young's inequality, we obtain for $\delta > 0$,

$$\begin{aligned} &- \int_\Omega u \int_{-\infty}^{+\infty} \eta(\xi) \phi(x,\xi,t) d\xi dx \\ &\ge \delta A_0 \|u\|^2 - \frac{1}{4\delta} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2 \, d\xi dx, \end{aligned}$$

which yields, by substitution in (43),

$$\begin{aligned} \varphi'(t) &\ge (1-\gamma) H^{-\gamma} H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 - \delta s \varepsilon \|u\|^2 \\ &\quad - \frac{b\varepsilon}{4\delta} \int_\Omega \int_{-\infty}^{+\infty} \left(\xi^2 + \beta\right) |\phi(x,\xi,t)|^2 \, d\xi dx + \varepsilon \|u\|_q^q. \end{aligned}$$

Using (39), we have

$$\varphi'(t) \geq \left[(1-\gamma) H^{-\gamma} - \frac{\varepsilon}{2\delta} \right] H'(t) \tag{44}$$
$$+ \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 - \delta s \varepsilon \|u\|^2 + \varepsilon \|u\|_q^q .$$

Therefore by taking δ so that $\frac{1}{2\delta} = kH^{-\gamma}(t)$, for large k to be specified later and substituting in (44), we arrive at

$$\varphi'(t) \geq \left[(1-\gamma) - \varepsilon k \right] H^{-\gamma}(t) H'(t) + \varepsilon \|u_t\|^2$$
$$- \varepsilon \|\Delta u\|^2 - \frac{s\varepsilon}{2k} H^{\gamma}(t) \|u\|_2^2 + \varepsilon \|u\|_q^q .$$

Consequently, using (38), we have for some $0 < r < 1$

$$\begin{aligned} \varphi'(t) \geq {} & \left[(1-\gamma) - \varepsilon k \right] H^{-\gamma}(t) H'(t) + \varepsilon \frac{q(1-r)+2}{2} \|u_t\|^2 \\ & + \varepsilon \frac{q(1-r)-2}{2} \|\Delta u\|^2 - \frac{s\varepsilon}{2k} H^{\gamma}(t) \|u\|^2 + \varepsilon r \|u\|_q^q \\ & + q(1-r)\varepsilon H(t) + \varepsilon \frac{q(1-r)b}{2} \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 \, d\xi dx \\ & + \varepsilon q(1-r) s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2 \, d\rho dx. \end{aligned} \tag{45}$$

By exploiting (40) and the inequality $\|u\| \leq C_* \|u\|_q$, we obtain

$$H^{\gamma}(t) \|u\|^2 \leq \left(\frac{1}{q} \right)^{\gamma} \|u\|_q^{q\gamma} \|u\|^2 \leq C_3 \|u\|^{q\gamma+2} .$$

Exploiting (42), we have

$$2 < q\gamma + 2 \leq q.$$

So, Lemma 11 yields

$$H^{\gamma}(t) \|u\|^2 \leq C_4 \left[\|\Delta u\|^2 + \|u\|_q^q \right] . \tag{46}$$

Inserting (46) in (45), we obtain

$$\begin{aligned} \varphi'(t) \geq {} & \left[(1-\gamma) - \varepsilon k \right] H^{-\gamma}(t) H'(t) + \varepsilon \frac{q(1-r)+2}{2} \|u_t\|^2 \\ & + \varepsilon \left[\frac{q(1-r)-2}{2} - \frac{C_4 s}{2k} \right] \|\Delta u\|_2^2 + \varepsilon \left[r - \frac{C_4 s}{2k} \right] \|u\|_q^q \\ & + q(1-r) \varepsilon H(t) + \varepsilon \frac{q(1-r)b}{2} \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 \, d\xi dx \\ & + \varepsilon q(1-r) s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2 \, d\rho dx. \end{aligned} \tag{47}$$

At this stage, we select a sufficiently small value for r, so that

$$q(1-r) - 2 > 0$$

and a sufficiently large value for k so that the following conditions hold

$$r - \frac{C_4 s}{2k} > 0, \qquad \frac{q(1-r)-2}{2} - \frac{C_4 s}{2k} > 0.$$

Given fixed values of r and k, we choose a sufficiently small ε such that

$$(1-\gamma) - \varepsilon k > 0, \qquad H(0) + \varepsilon \int_\Omega u_0 u_1 dx > 0.$$

So, (47) becomes, for some $C_5 > 0$

$$\varphi'(t) \geq C_5 \Big[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \|u\|_q^q + b \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + s\tau \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \Big]$$

and

$$\varphi(t) \geq \varphi(0) > 0, \quad t \geq 0.$$

Conversely, applying Hölder's inequality together with the embedding inequality $L\|u\| \leq C_* \|u\|_q$, we have

$$\int_\Omega u u_t dx \leq \|u\| \|u_t\| \leq C_* \|u\|_q \|u_t\|.$$

By applying Young's inequality alongside Lemma 11, it follows that

$$\left(\int_\Omega u u_t dx\right)^{\frac{1}{1-\gamma}} \leq C_6 \left[\|u\|_q^l + \|u_t\|^2\right] \leq C_7 \left[\|u_t\|^2 + \|\Delta u\|^2 + \|u\|_q^q\right], \tag{48}$$

where C_6 and C_7 are positive constants and $2 \leq l = \frac{1}{1-2\gamma} \leq q$. Therefore,

$$\varphi^{\frac{1}{1-\gamma}}(t) \leq 2^{\frac{1}{1-\gamma}} \left[H(t) + \left(\int_\Omega u u_t dx\right)^{\frac{1}{1-\gamma}}\right] \leq C_8 \left[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \|u\|_q^q\right], \quad t \geq 0, \tag{49}$$

where C_8 is a positive constant. Combining (48) and (49), we arrive at

$$\varphi'(t) \geq C_9 \varphi^{\frac{1}{1-\gamma}}(t), \quad t \geq 0. \tag{50}$$

A simple integration of (50) over $(0,t)$, we get

$$\varphi(t) \geq \left(\frac{1}{\varphi^{\frac{-\gamma}{1-\gamma}}(0) - \frac{\gamma}{1-\gamma} C_9 t}\right)^{\frac{1-\gamma}{\gamma}}.$$

So, $\varphi(t)$ blows up in time

$$T \leq T^* = \frac{1-\gamma}{C_9 \gamma \varphi^{\frac{1}{1-\gamma}}(0)}.$$

The proof is completed. □

Conclusion

In recent years, considerable attention has been given to the wave equation with fractional time delays. However, to the best of our knowledge, no studies have addressed the existence, decay, and blow-up of solutions for the Petrovsky equation incorporating a fractional time delay. In this paper, under appropriate assumptions, we establish results concerning the existence, decay, and blow-up of solutions within a bounded domain.

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